(iii) Relation between E and δ:

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

$$= E^{1/2} f(x) - E^{-1/2} f(x)$$

$$\delta f(x) = \left[E^{1/2} - E^{-1/2}\right] f(x)$$

$$\therefore \boxed{\delta = E^{1/2} - E^{-1/2}} \rightarrow (1) \quad \text{(or)} \quad \delta = E^{-1/2}(E-1) \Rightarrow \boxed{\delta = \Delta E^{-1/2}}$$

Also from (1), we have $\delta = E^{1/2}\left[1 - E^{-1}\right]$

$$\boxed{\delta = E^{1/2}\nabla}$$

(iv) Relation between E and μ

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

$$= \frac{1}{2}\left[E^{1/2} f(x) + E^{-1/2} f(x)\right]$$

$$= \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] f(x)$$

$$\therefore \boxed{\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]}$$

(v) Relation between D and Δ.

$$D.\, f(x) = \frac{d}{dx} f(x).$$

By Taylor's theorem,

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \cdots$$

$$\therefore E f(x) = f(x) + \frac{h}{1!} D f(x) + \frac{h^2}{2!} D^2 f(x) + \cdots$$

$$= \left[1 + \frac{hD}{1!} + \frac{(hD)^2}{2!} + \cdots\right] f(x)$$

$$= e^{hD} \cdot f(x)$$

$$\therefore \boxed{E = e^{hD}}$$

$$\therefore E = 1 + \Delta = e^{hD}$$

$$hD = \log E = \log(1+\Delta)$$

$$hD = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \cdots$$

$$\therefore D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \cdots\right]$$

Forward Difference Table :

The finite forward differences of a functions are represented below in a tabular form.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

The above table is also called diagonal difference table.

Note : 1. The value y_0 (first value of y) is called the leading term and the differences $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0 \ldots$ are called the leading differences.

2. $y_1 \to \Delta y_1$

$\uparrow$

y_2

The ~~difference~~ difference value $y_2 - y_1$ is written in the next column in between y_2 and y_1.

Backward Difference table:

The backward differences are given in the following backward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_0	y_0						
		∇y_1					
x_1	y_1		$\nabla^2 y_2$				
				$\nabla^3 y_3$			
		∇y_2					
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$		
						$\nabla^5 y_5$	
		∇y_3		$\nabla^3 y_4$			
x_3	y_3						$\nabla^6 y_6$
			$\nabla^2 y_4$		$\nabla^4 y_5$		
		∇y_4		$\nabla^3 y_5$		$\nabla^5 y_6$	
x_4	y_4						
			$\nabla^2 y_5$		$\nabla^4 y_5$		
		∇y_5		$\nabla^3 y_6$			
x_5	y_5						
			$\nabla^2 y_6$				
		∇y_6					
x_6	y_6						

Note:

1. In this table y_6, the value of y is the leading term and ∇y_6, $\nabla^2 y_6$, $\nabla^3 y_6, \ldots$ are leading differences. These leading differences lie along the diagonal sloping upwards at the end.

Problem:

1) Find the seventh term of the sequence 4, 15, 40, 85, 156 & 259.

Sol: Formula to find y_k in terms of y_0, Δy_0, $\Delta^2 y_0, \ldots$

i.e., $y_k = E^k y_0$

$= (1+\Delta)^k y_0$

$= \left[1 + \binom{k}{1}\Delta + \binom{k}{2}\Delta^2 + \binom{k}{3}\Delta^3 + \dots\right] y_0$

$y_k = y_0 + \binom{k}{1}\Delta y_0 + \binom{k}{2}\Delta y_0 + \binom{k}{3}\Delta y_0 + \dots + \Delta^k y_0$

7^{th} term, $y_6 = y_0 + \binom{6}{1}\Delta y_0 + \binom{6}{2}\Delta^2 y_0 + \binom{6}{3}\Delta^3 y_0 + \binom{6}{4}\Delta^4 y_0$
$+ \binom{6}{5}\Delta^2 y_0 + \binom{6}{6}\Delta^6 y_0.$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	4					
		11				
1	15		14			
		25		6		
2	40		20		0	
		45		6		0
3	85		26		0	
		71		6		
4	156		32			
		103				
5	259					

We know that, $y_k = E^k y_0 = (1+\Delta)^k y_0$

$y_6 = y(x=7) = y_0 + (6C_1)\Delta y_0 + (6C_2)\Delta^2 y_0 + \dots + \Delta^6 y_0.$

$= 4 + 6(11) + 15(14) + 20(6) + 15(0) + 0 + 0$

$= 4 + 66 + 210 + 120$

$\boxed{y_6 = 400}$

2. Find the general term and 7^{th} term of the sequence 2, 9, 28, 65, 126, 217.

Sol:

General term,

$$y_n = E^n y_0$$

$$= (1+\Delta)^n y_0$$

$$= y_0 + {}^nC_1 \Delta y_0 + {}^nC_2 \Delta^2 y_0 + {}^nC_3 \Delta^3 y_0 + \cdots$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	2				
		7			
1	9		12		
		19		6	
2	28		18		0
		37		6	
3	65		24		0
		61		6	
4	126		30		
		91			
5	217				

i.e., $y_n = 2 + n(7) + {}^nC_2\, 12 + {}^nC_3(6) + 0$

$$= 2 + 7n + \frac{n(n-1)}{2}(12) + \frac{n(n-1)(n-2)}{3\times2\times1}(6) + 0$$

$$= 2 + 7n + 6n^2 - 6n + n^3 - 2n^2 - n^2 + 2n$$

$$= n^3 + 3n^2 + 3n + 1 + 1$$

7^{th} term, $y_6 = 6^3 + 3(6^2) + 3(6) + 1 + 1 = 344$

3. Find $f(x)$ from the table below. Also find $f(7)$.

x :	0	1	2	3	4	5	6
$f(x)$:	-1	3	19	53	111	199	323.

Sol:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	-1	4	12	6	0
1	3	16	18	6	0
2	19	34	24	6	0
3	53	58	30	6	
4	111	88	36		
5	199	124			
6	323				

etc.

$$y_x = E^x y_0 = (1+\Delta)^x y_0$$

$$= y_0 + {}^xC_1 \Delta y_0 + {}^xC_2 \Delta^2 y_0 + {}^xC_3 \Delta^3 y_0 + {}^xC_4 \Delta^4 y_0 + \cdots$$

$$= -1 + x(4) + \frac{x(x-1)}{2}(12) + \frac{x(x-1)(x-2)}{3\times 2}(6) + 0$$

$$= -1 + 4x + 6x^2 - 6x + x^3 - 3x^2 + 2x$$

$$f(x) = x^3 + 3x^2 - 1$$

$$f(7) = 7^3 + 3(49) - 1 = 489$$

4. Express any value of y in terms of y_n and the backward difference of y_n.

We know,

$$\nabla y_n = y_n - y_{n-1}$$

$$\therefore \quad y_{n-1} = y_n - \nabla y_n$$

$$\boxed{y_{n-1} = (1-\nabla) y_n}$$

|||ly

$$y_{n-2} = y_{n-1} - \nabla y_{n-1}$$

$$y_{n-2} = (1-\nabla) y_{n-1}$$

$$= (1-\nabla)(1-\nabla) y_n$$

$$y_{n-2} = (1-\nabla)^2 y_n \quad \text{and}$$

$$y_{n-3} = (1-\nabla)^3 y_n$$